Shallowly Embedding Type Theories in Agda as Presheaf Models

Work in Progress

Joris Ceulemans   Dominique Devriese

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https://github.com/JorisCeulemans/shallow-presheaf-embedding

Introduction

- Framework for exploring type theories with presheaf models in Agda.
- Manipulating contexts, types, terms, ... of the object theory.
- Worked out example: guarded recursion.

Guarded Recursion: Quick Introduction/Reminder

$\mathsf{iterate} : (A \to A) \to A \to \mathsf{Stream}\ A$
$\mathsf{head}\ (\mathsf{iterate}\ f\ a) = a$
$\mathsf{tail}\ (\mathsf{iterate}\ f\ a) = \mathsf{map}\ f\ (\mathsf{iterate}\ f\ a)$

iterate is productive but not accepted by termination checker.

Possible solutions:

- Sized types,
- Guarded recursion.

Guarded Recursion: Quick Introduction/Reminder

Later modality:

$$\frac{\vdash A \text{ type}}{\vdash \triangleright A \text{ type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{next}\, a : \triangleright A} \qquad \frac{\Gamma \vdash f : \triangleright(A \to B) \quad \Gamma \vdash a : \triangleright A}{\Gamma \vdash f \circledast a : \triangleright B}$$

Löb induction:

$$\frac{\Gamma \vdash f : \triangleright A \to A}{\Gamma \vdash \mathsf{löb}\, f : A} \qquad f\left(\mathsf{next}\,(\mathsf{löb}\, f)\right) \approx \mathsf{löb}\, f$$

Guarded streams:

$$\frac{\Gamma \vdash as : \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{head}\, as : A} \qquad \frac{\Gamma \vdash as : \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{tail}\, as : \triangleright \mathsf{Stream}\, A}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash as : \triangleright \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{cons}\, a\, as : \mathsf{Stream}\, A}$$

Guarded Recursion: Quick Introduction/Reminder

$$\frac{\vdash A \text{ type}}{\vdash \triangleright A \text{ type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{next}\, a : \triangleright A} \qquad \frac{\Gamma \vdash f : \triangleright(A \to B) \quad \Gamma \vdash a : \triangleright A}{\Gamma \vdash f \circledast a : \triangleright B}$$

$$\frac{\Gamma \vdash f : \triangleright A \to A}{\Gamma \vdash \text{löb}\, f : A} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash as : \triangleright \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{cons}\, a\, as : \mathsf{Stream}\, A}$$

If $f : A \to A$, then $\mathsf{map}\, f : \mathsf{Stream}\, A \to \mathsf{Stream}\, A$.

$$\mathsf{iterate}\, f\, a = \text{löb}\,\Big(\qquad \Big)$$

Guarded Recursion: Quick Introduction/Reminder

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$$\frac{\Gamma \vdash f : \triangleright A \to A}{\Gamma \vdash \mathsf{l\ddot{o}b}\, f : A} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash as : \triangleright \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{cons}\, a\, as : \mathsf{Stream}\, A}$$

If $f : A \to A$, then $\mathsf{map}\, f : \mathsf{Stream}\, A \to \mathsf{Stream}\, A$.

$$\mathsf{iterate}\, f\, a = \mathsf{l\ddot{o}b}\left(\lambda\, x : \triangleright \mathsf{Stream}\, A\, . \qquad\qquad\qquad\qquad \right)$$

Guarded Recursion: Quick Introduction/Reminder

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Guarded Recursion: Quick Introduction/Reminder

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If $f : A \to A$, then $\mathsf{map}\, f : \mathsf{Stream}\, A \to \mathsf{Stream}\, A$.

$$\mathsf{iterate}\, f\, a = \mathsf{l\ddot{o}b}\left(\lambda x : \triangleright \mathsf{Stream}\, A \,.\, \mathsf{cons}\, a\, (\qquad (\mathsf{map}\, f) \qquad x)\right)$$

Guarded Recursion: Quick Introduction/Reminder

$$\frac{\vdash A \text{ type}}{\vdash \triangleright A \text{ type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{next}\, a : \triangleright A} \qquad \frac{\Gamma \vdash f : \triangleright(A \to B) \quad \Gamma \vdash a : \triangleright A}{\Gamma \vdash f \circledast a : \triangleright B}$$

$$\frac{\Gamma \vdash f : \triangleright A \to A}{\Gamma \vdash \mathsf{löb}\, f : A} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash as : \triangleright \mathsf{Stream}\, A}{\Gamma \vdash \mathsf{cons}\, a\, as : \mathsf{Stream}\, A}$$

If $f : A \to A$, then $\mathsf{map}\, f : \mathsf{Stream}\, A \to \mathsf{Stream}\, A$.

$$\mathsf{iterate}\, f\, a = \mathsf{löb}\left(\lambda x : \triangleright \mathsf{Stream}\, A \,.\, \mathsf{cons}\, a\, (\mathsf{next}\, (\mathsf{map}\, f) \circledast x)\right)$$

A Preview: Exploring Guarded Recusion Using Agda

$\triangleright$' : $\{\Gamma : \text{Ctx}\ \omega\ \ell\} \to \text{Ty}\ \Gamma \to \text{Ty}\ \Gamma$
$\triangleright$' $\{\Gamma = \Gamma\}\ T = \ldots$

next' : $\{\Gamma : \text{Ctx}\ \omega\ \ell\}\ \{T : \text{Ty}\ \Gamma\} \to \text{Tm}\ \Gamma\ T \to \text{Tm}\ \Gamma\ (\triangleright\text{'}\ T)$
next' $t = \ldots$

löb : $\{\Gamma : \text{Ctx}\ \omega\ \ell\}\ (T : \text{Ty}\ \Gamma) \to \text{Tm}\ \Gamma\ (\triangleright\text{'}\ T \Rrightarrow T) \to \text{Tm}\ \Gamma\ T$
löb $T\ f = \ldots$

Stream : $\{\Gamma : \text{Ctx}\ \omega\ 0\ell\} \to \text{Ty}\ \Gamma$
Stream $\{\Gamma = \Gamma\} = \ldots$
$\rightsquigarrow$ guarded streams of natural numbers

Guarded Recursion in the Topos of Trees

Representation for Stream $\mathbb{N}$:

$$\mathsf{Vec}_1\,\mathbb{N} \longleftarrow \mathsf{Vec}_2\,\mathbb{N} \longleftarrow \ldots \longleftarrow \mathsf{Vec}_n\,\mathbb{N} \longleftarrow \mathsf{Vec}_{n+1}\,\mathbb{N} \longleftarrow \ldots$$

And for a general closed type X:

$$X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_n \longleftarrow X_{n+1} \longleftarrow \ldots$$

Guarded Recursion in the Topos of Trees

Representation for Stream $\mathbb{N}$:

$$\mathsf{Vec}_1\,\mathbb{N} \longleftarrow \mathsf{Vec}_2\,\mathbb{N} \longleftarrow \ldots \longleftarrow \mathsf{Vec}_n\,\mathbb{N} \longleftarrow \mathsf{Vec}_{n+1}\,\mathbb{N} \longleftarrow \ldots$$

And for a general closed type X (and $\triangleright X$):

$$X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_n \longleftarrow X_{n+1} \longleftarrow \ldots$$

$$\top \longleftarrow X_0 \longleftarrow \ldots \longleftarrow X_{n-1} \longleftarrow X_n \longleftarrow \ldots$$

Guarded Recursion in the Topos of Trees

Representation for Stream $\mathbb{N}$:

$$\mathsf{Vec}_1\,\mathbb{N} \longleftarrow \mathsf{Vec}_2\,\mathbb{N} \longleftarrow \ldots \longleftarrow \mathsf{Vec}_n\,\mathbb{N} \longleftarrow \mathsf{Vec}_{n+1}\,\mathbb{N} \longleftarrow \ldots$$

And for a general closed type X (and $\triangleright X$):

$$\begin{tikzcd}
X_0 & X_1 \arrow[l] & \ldots \arrow[l] & X_n \arrow[l] & X_{n+1} \arrow[l] & \ldots \arrow[l] \\
\top \arrow[u] & X_0 \arrow[l] \arrow[u] & \ldots \arrow[l] & X_{n-1} \arrow[l] \arrow[u] & X_n \arrow[l] \arrow[u] & \ldots \arrow[l]
\end{tikzcd}$$

Guarded Recursion in the Topos of Trees

Representation for $\mathsf{Stream}\,\mathbb{N}$:

$$\mathsf{Vec}_1\,\mathbb{N} \longleftarrow \mathsf{Vec}_2\,\mathbb{N} \longleftarrow \dots \longleftarrow \mathsf{Vec}_n\,\mathbb{N} \longleftarrow \mathsf{Vec}_{n+1}\,\mathbb{N} \longleftarrow \dots$$

And for a general closed type X (and $\triangleright X$):

$$\begin{tikzcd}
X_0 & X_1 \arrow[l] & \dots \arrow[l] & X_n \arrow[l] & X_{n+1} \arrow[l] & \dots \arrow[l] \\
\top \arrow[u] & X_0 \arrow[l] \arrow[u] & \dots \arrow[l] & X_{n-1} \arrow[l] \arrow[u] & X_n \arrow[l] \arrow[u] & \dots \arrow[l]
\end{tikzcd}$$

What about contexts?

$$x : \mathsf{Stream}\,\mathbb{N} \vdash \dots$$

$\Rightarrow$ Same representation.

Guarded Recursion in the Topos of Trees

Equivalent formulation:

$$\begin{tikzcd}
X_0 \arrow[loop below, "\mathrm{id}"] & X_1 \arrow[l] \arrow[loop below, "\mathrm{id}"] & \ldots \arrow[l] & X_n \arrow[l] \arrow[ll, bend right] \arrow[lll, bend right] \arrow[loop below, "\mathrm{id}"] & \ldots \arrow[l]
\end{tikzcd}$$

Internal Cwf Structure: Contexts, Types, Terms, Substitutions

record Ctx ℓ : Set (lsuc ℓ) where ...

$\diamond$: Ctx ℓ

record Ty $\{\ell\}$ (Γ : Ctx ℓ) : Set (lsuc ℓ) where ...

record Tm $\{\ell\}$ (Γ : Ctx ℓ) (T : Ty Γ) : Set ℓ where ...

record _⇒_ $\{\ell\}$ (Δ Γ : Ctx ℓ) : Set ℓ where ...

_[_] : Ty $\Gamma \to \Delta \Rightarrow \Gamma \to$ Ty Δ

_[_]' : Tm Γ $T \to (\sigma : \Delta \Rightarrow \Gamma) \to$ Tm Δ (T [σ])

Internal Cwf Structure: Context Extension, Equivalence

$\_,,\_ : (\Gamma : \mathsf{Ctx}\ \ell)\ (T : \mathsf{Ty}\ \Gamma) \to \mathsf{Ctx}\ \ell$

$\pi : \Gamma\ ,,\ T \Rightarrow \Gamma$

$\xi : \mathsf{Tm}\ (\Gamma\ ,,\ T)\ (T\ [\ \pi\ ])$

$\rightsquigarrow$ corresponds to judgement $\Gamma, x : T \vdash x : T$.

record $\_\cong^{\mathsf{ty}}\_\ \{\ell\}\ \{\Gamma : \mathsf{Ctx}\ \ell\}\ (T\ S : \mathsf{Ty}\ \Gamma) : \mathsf{Set}\ \ell$ where ...

record $\_\cong^{\mathsf{tm}}\_\ \{\Gamma : \mathsf{Ctx}\ \ell\}\ \{T : \mathsf{Ty}\ \Gamma\}\ (t\ s : \mathsf{Tm}\ \Gamma\ T) : \mathsf{Set}\ \ell$ where ...

$\iota[\_]\_ : T \cong^{\mathsf{ty}} S \to \mathsf{Tm}\ \Gamma\ S \to \mathsf{Tm}\ \Gamma\ T$

Simple Types

$$\text{Nat'} : \text{Ty}\ \Gamma$$
$$\text{zero'} : \text{Tm}\ \Gamma\ \text{Nat'}$$
$$\text{suc'} : \text{Tm}\ \Gamma\ \text{Nat'} \to \text{Tm}\ \Gamma\ \text{Nat'}$$

$$\_\boxtimes\_ : \text{Ty}\ \Gamma \to \text{Ty}\ \Gamma \to \text{Ty}\ \Gamma$$
$$\text{pair} : \text{Tm}\ \Gamma\ T \to \text{Tm}\ \Gamma\ S \to \text{Tm}\ \Gamma\ (T \boxtimes S)$$
$$\text{fst} : \text{Tm}\ \Gamma\ (T \boxtimes S) \to \text{Tm}\ \Gamma\ T$$
$$\text{snd} : \text{Tm}\ \Gamma\ (T \boxtimes S) \to \text{Tm}\ \Gamma\ S$$

$$\_\Rrightarrow\_ : \text{Ty}\ \Gamma \to \text{Ty}\ \Gamma \to \text{Ty}\ \Gamma$$
$$\text{lam} : (T : \text{Ty}\ \Gamma) \to \text{Tm}\ (\Gamma\ ,,\ T)\ (S\ [\ \pi\ ]) \to \text{Tm}\ \Gamma\ (T \Rrightarrow S)$$
$$\text{app} : \text{Tm}\ \Gamma\ (T \Rrightarrow S) \to \text{Tm}\ \Gamma\ T \to \text{Tm}\ \Gamma\ S$$

Later Modality, Löb Induction, Guarded Streams

$$\triangleright\text{'} : \mathsf{Ty}\ \Gamma \to \mathsf{Ty}\ \Gamma$$
$$\mathsf{next'} : \mathsf{Tm}\ \Gamma\ T \to \mathsf{Tm}\ \Gamma\ (\triangleright\text{'}\ T)$$

$$\mathsf{löb} : (T : \mathsf{Ty}\ \Gamma) \to \mathsf{Tm}\ \Gamma\ (\triangleright\text{'}\ T \Rrightarrow T) \to \mathsf{Tm}\ \Gamma\ T$$
$$\mathsf{löb\text{-}is\text{-}fixpoint} : (f : \mathsf{Tm}\ \Gamma\ (\triangleright\text{'}\ T \Rrightarrow T)) \to$$
$$\mathsf{löb}\ T\ f \cong^{\mathsf{tm}} \mathsf{app}\ f\ (\mathsf{next'}\ (\mathsf{löb}\ T\ f))$$

$$\mathsf{Stream} : \{\Gamma : \mathsf{Ctx}\ 0\ell\} \to \mathsf{Ty}\ \Gamma$$
$$\mathsf{str\text{-}head} : \mathsf{Tm}\ \Gamma\ \mathsf{Stream} \to \mathsf{Tm}\ \Gamma\ \mathsf{Nat'}$$
$$\mathsf{str\text{-}tail} : \mathsf{Tm}\ \Gamma\ \mathsf{Stream} \to \mathsf{Tm}\ \Gamma\ (\triangleright\ \mathsf{Stream})$$
$$\mathsf{str\text{-}cons} : \mathsf{Tm}\ \Gamma\ (\mathsf{Nat'} \boxtimes (\triangleright\ \mathsf{Stream})) \to \mathsf{Tm}\ \Gamma\ \mathsf{Stream}$$

$$\mathsf{zeros} : \mathsf{Tm}\ \diamond\ \mathsf{Stream}$$

General Base Categories

Recall: Representation of a closed type

$$X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_n \longleftarrow X_{n+1} \longleftarrow$$

Diagrams of other shape possible (and useful).

$\Rightarrow$ Closed type = presheaf over certain base category.

Only change necessary (for user): Ctx C ℓ instead of Ctx ℓ.

Closely Related Work

- In Agda:
  - Niccolò Veltri, Niels van der Weide: Guarded Recursion in Agda via Sized Types
  - Casper Bach Poulsen, Arjen Rouvoet, Andrew Tolmach, Robbert Krebbers, Eelco Visser: `https://github.com/metaborg/mj.agda/tree/develop` accompanying "Intrinsically-Typed Definitional Interpreters for Imperative Languages"
- In Coq:
  - Guilhem Jaber, Nicolas Tabareau, Matthieu Sozeau: Extending Type Theory with Forcing
- In Nuprl:
  - Mark Bickford: Formalizing Category Theory and Presheaf Models of Type Theory in Nuprl

Any others?

Future Work

- Make the system more user-friendly.
  - E.g. proof by reflection (tactics) for proving type equality.
- Explore several non-dependent type theories.
  - E.g. nominal sets, very restricted form of parametricity, normalization using presheaves, ...
  - Other ideas?
- Add full dependent types.
  - Types can already depend on variables in context.
  - Universe types seem most complicated to add.