Chapter 2

Higher Order Programming, Data Structures & Data Abstraction

Previously

We concentrated on computational processes and the role of procedures in program design

- **primitive data** (numbers) and **primitive operations** (arithmetic)
- **combination** of procedures to form **compound procedures** through composition, conditionals, and the use of parameters
- **abstraction** of procedures by using define
- **processes** generated by procedures
- **higher-order procedures** to enhance the power of our language by enabling us to manipulate, and thereby to reason in terms of, general methods of computation

Status So Far

	data	procedures
primitive	`0, 1, -3.14, …`	`+, -, *, …`
combinations		`(lambda (x) ...)`
abstraction		`(define f` `  (lambda (x) ...)`

Now we treat compound data

Why Compound Data?

- elevate the conceptual level
- increase the modularity
- enhance the expressive power

Overal goal: structure our programs so that they can operate on abstract data

Elevate Conceptual Level

A rational number can be thought of as a numerator and a denominator (i.e. 2 numbers) or as "a rational number" (i.e. 1 number that happens to consist of 2 numbers)

E.g., don't represent rationals just by two separate integers + pairs of separate procedures to compute operations (one for nominator and one for denominator)

Increase Modularity

Separate the part of our program that deals with rational numbers per se from the details of how rational numbers may be represented as pairs of integers.

The part of the program using rational numbers does not (should not) be aware of how rational numbers are defined!

Data abstraction

Increase in Expressive Power

```
(define (linear-combination a b x y)
  (+ (* a x) (* b y)))
```

Only for "simple" numbers

```
(define (linear-combination a b x y)
  (add (mul a x) (mul b y)))
```

Requires the ability for direct manipulation of compound objects

Also for rational numbers, complex numbers, polynomials, ...

Illustration: Rational Numbers

Suppose we're writing a mathematical system.

```
(make-rat <n> <d>)
(numer <r>)
(denom <r>)
```

Suppose we had these

Then this is possible!

```
(define (add-rat x y)
  (make-rat (+ (* (numer x) (denom y))
               (* (numer y) (denom x)))
            (* (denom x) (denom y))))
(define (sub-rat x y)
  (make-rat (- (* (numer x) (denom y))
               (* (numer y) (denom x)))
            (* (denom x) (denom y))))
(define (mul-rat x y)
  (make-rat (* (numer x) (numer y))
            (* (denom x) (denom y))))
(define (div-rat x y)
  (make-rat (* (numer x) (denom y))
            (* (denom x) (numer y))))
```

Now let's glue together numer and denom...

Structuring Data in Scheme

Axiom

```
(car (cons x y)) = x
(cdr (cons x y)) = y
```

```
> (define x (cons 1 2))
> (car x)
1
> (cdr x)
2
```

```
> (define x (cons 1 2))
> (define y (cons 3 4))
> (define z (cons x y))
> (car (car z))
1
> (car (cdr z))
3
```

Any structure can be made

Back to our Rational Numbers

4 alternative implementations

```
(define make-rat cons)
(define numer car)
(define denom cdr)
```

efficient but less debuggable

```
(define (make-rat n d)
  (cons n d))
(define (numer r)
  (car r))
(define (denom r)
  (cdr r))
```

```
(define (make-rat n d)
  (let ((g (gcd n d)))
    (cons (/ n g) (/ d g))))
(define (numer r)
  (car r))
(define (denom r)
  (cdr r))
```

```
(define (make-rat n d)
  (cons n d))
(define (numer x)
  (let ((g (gcd (car x) (cdr x))))
    (/ (car x) g)))
(define (denom x)
  (let ((g (gcd (car x) (cdr x))))
    (/ (cdr x) g)))
```

Interludium: Textual Output

```
(display <expression>)
(newline)
```

```
(define (print-rat x)
  (display (numer x))
  (display "/")
  (display (denom x))
  (newline))
```

A body with multiple expressions

What is meant by Data ?

Not any three procedures result in rat nums

`make-rat`, `numer`, and `denom` must satisfy the condition that,
for any integer `n` and any non-zero integer `d`,
if `x` is `(make-rat n d)`, then

$$\frac{\texttt{(numer x)}}{\texttt{(denom x)}} = \frac{\texttt{n}}{\texttt{d}}$$

data = constructors and selectors + conditions

Constructing Data Only Using Procedures

```
(car (cons x y)) = x
(cdr (cons x y)) = y
```

Procedural representation of cons cells

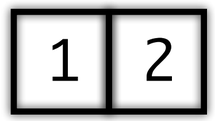
```
(define (cons x y)
  (define (dispatch m)
    (cond ((= m 0) x)
          ((= m 1) y)
          (else (error "Argument not 0 or 1 -- cons" m))))
  dispatch)

(define (car z)
  (z 0))
(define (cdr z)
  (z 1))
```

This curiosity will form the basis of object-oriented programming in Scheme (c.f. chapter 3)

This “construction of pairs using mathematical functions” was invented in 1934 by Alonzo Church (λ-calculus)

Box-and-pointer Diagrams

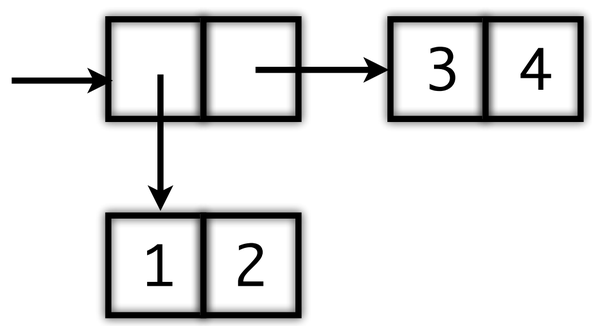
```
(cons 1 2)
```



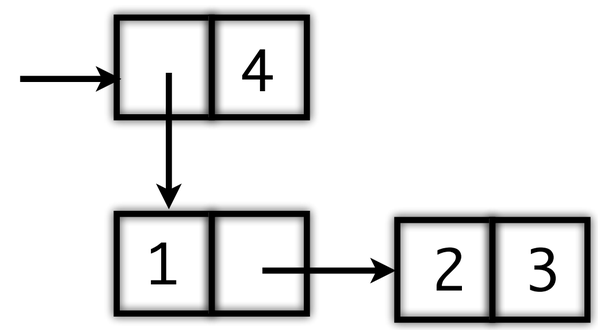
A pair's elements
can be pairs again.

Closure property for pairs

```
(cons (cons 1 2)
      (cons 3 4))
```



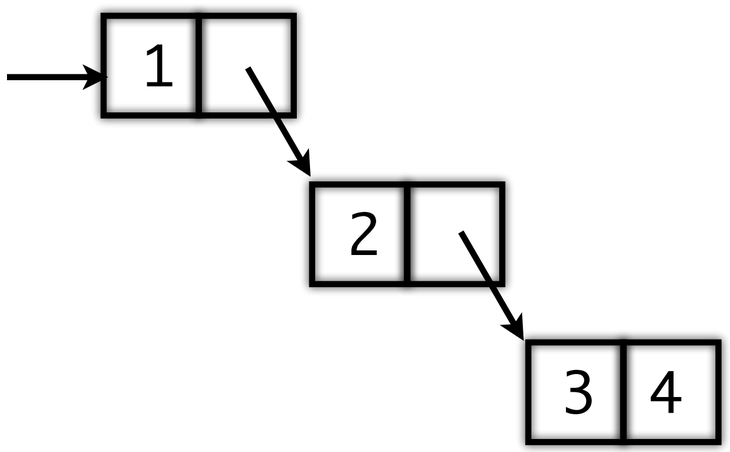

```
(cons (cons 1
            (cons 2 3))
      4)
```



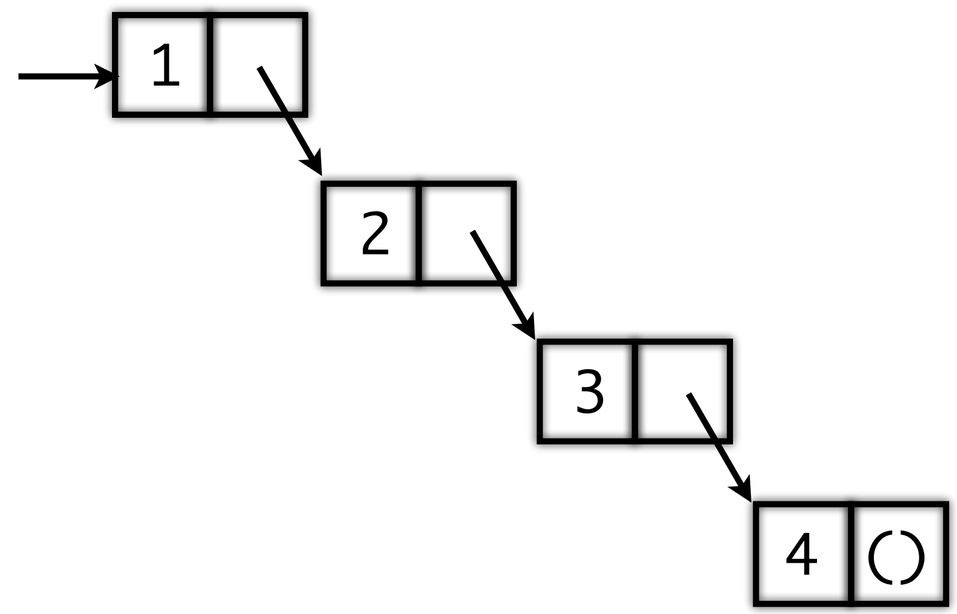
Lists : An Idiom of Using Pairs

A list is

- either the empty list '()
- any pair whose cdr is a list



not a list!



a list

For your convenience

```
> (list 1 2 3 4)
(1 2 3 4)
> (cons 1 (cons 2 (cons 3 (cons 4 '()))))
(1 2 3 4)
> (caddr (list 1 2 3 4))
3
> (caar (cons (cons 1 2) (cons 3 4)))
1
> (null? '())
#t
> (null? (list))
#t
> (null? (list 1))
#f
> (null? (cons 1 2))
#f
```

A Very Common Pitfall

```
> (1 2 3 4 5)
⊕ procedure application: expected procedure, given: 1; arguments
were: 2 3 4 5
>
```

Remember the evaluation rule for combinations

List Operations (1)

Recursive Processes

```
(define (list-ref items n)
  (if (= n 0)
      (car items)
      (list-ref (cdr items) (- n 1))))

(define (length items)
  (if (null? items)
      0
      (+ 1 (length (cdr items)))))
```

The `length` of the empty list is `0`

The `length` of any list is 1 plus the `length` of the `car` of the list

"`cdr`ing down" lists

```
> (define squares (list 1 4 9 16 25))
> (list-ref squares 3)
16
> (define odds (list 1 3 5 7))
> (length odds)
4
```

List Operations (2)

```
(define (length items)
  (define (length-iter a count)
    (if (null? a)
        count
        (length-iter (cdr a) (+ 1 count))))
  (length-iter items 0))

(define (append list1 list2)
  (if (null? list1)
      list2
      (cons (car list1) (append (cdr list1) list2))))
```

Iterative Process

```
> (append squares odds)
(1 4 9 16 25 1 3 5 7)
> (append odds squares)
(1 3 5 7 1 4 9 16 25)
```

Higher Order Procedures

```
(define (map proc items)
  (if (null? items)
      '()
      (cons (proc (car items))
            (map proc (cdr items)))))
```

Abstraction!
(no cons,car,cdr)

```
> (map abs (list -10 2.5 -11.6 17))
(10 2.5 11.6 17)
> (map (lambda (x) (* x x))
       (list 1 2 3 4))
(1 4 9 16)
> (define (scale-list items factor)
    (map (lambda (x) (* x factor))
         items))
> (scale-list (list 1 2 3 4 5) 10)
(10 20 30 40 50)
```

Lambdas on steroids

apply takes a procedure and a list of arguments (both can be computed dynamically) and applies the procedure to the arguments

```
> (define f (lambda (x y . rest)
              (if (null? rest)
                  (+ x y)
                  (+ x y (car rest)))))
> (f 1 2)
3
> (f 1 2 3)
6
> (f 1 2 3 4)
6
> (f 1)
f: arity mismatch;
```

```
> (define sum-all (lambda lst
                    (accumulate + 0 lst)))
> (sum-all 1 2 3 4)
10
> (define sum-all2 (lambda (lst)
                    (accumulate + 0 lst)))
> (sum-all2 1 2 3 4)
sum-all2: arity mismatch;
> (sum-all2 (list 1 2 3 4))
10
> (apply sum-all2 (list 1 2 3 4))
sum-all2: arity mismatch;
> (apply sum-all (list 1 2 3 4))
10
```

Trees

A tree is a list whose elements are lists. The elements of the list are the branches of the tree.

```
(define (count-leaves x)
  (cond ((null? x) 0)
        ((not (pair? x)) 1)
        (else (+ (count-leaves (car x))
                 (count-leaves (cdr x))))))
```

Tree-recursive process

```
> (define x (cons (list 1 2) (list 3 4)))
> (length x)
3
> (count-leaves x)
4
> (length (list x x))
2
> (count-leaves (list x x))
8
```

Lists as Conventional Interfaces

Previous: Data abstraction allows us to "forget" about the internal (concrete) representation of data

Flexibility to experiment with alternative representations

New principle: Use list as conventional interface

Convert compound data into lists + apply list operators

Motivation

```
(define (sum-odd-squares tree)
  (cond ((null? tree) 0)
        ((not (pair? tree))
         (if (odd? tree) (square tree) 0))
        (else (+ (sum-odd-squares (car tree))
                 (sum-odd-squares (cdr tree))))))
```

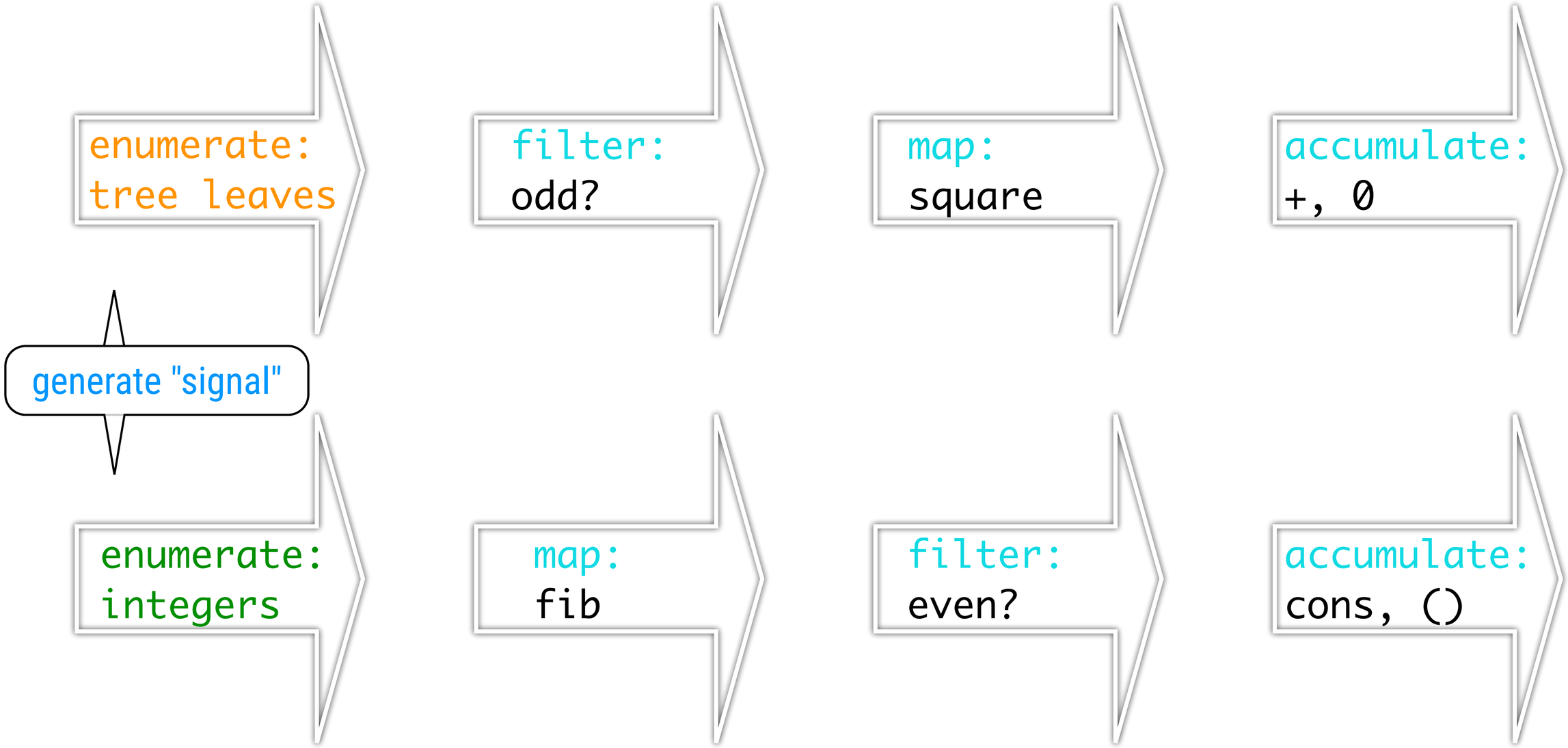
Compute the sum of the squares of the leaves that are odd

A list of all even fibonacci numbers f_k with $k \leq n$

```
(define (even-fibs n)
  (define (next k)
    (if (> k n)
        '()
        (let ((f (fib k)))
          (if (even? f)
              (cons f (next (+ k 1)))
              (next (+ k 1))))))
  (next 0))
```

These procedures *seem* very different

A Signal-Processing Engineer's View



Let's make this structure explicit in the code...

Higher Order List Procedures

```
(define (filter predicate sequence)
  (cond ((null? sequence) '())
        ((predicate (car sequence))
         (cons (car sequence)
               (filter predicate (cdr sequence))))
        (else (filter predicate (cdr sequence)))))

(define (accumulate op initial sequence)
  (if (null? sequence)
      initial
      (op (car sequence)
          (accumulate op initial (cdr sequence)))))
```

```
> (filter odd? (list 1 2 3 4 5))
(1 3 5)
> (accumulate + 0 (list 1 2 3 4 5))
15
> (accumulate * 1 (list 1 2 3 4 5))
120
> (accumulate cons '() (list 1 2 3 4 5))
(1 2 3 4 5)
```

Lists as Conventional Interfaces (1)

```
(define (enumerate-interval low high)
  (if (> low high)
      '()
      (cons low (enumerate-interval (+ low 1) high))))

(define (enumerate-tree tree)
  (cond ((null? tree) '())
        ((not (pair? tree)) (list tree))
        (else (append (enumerate-tree (car tree))
                      (enumerate-tree (cdr tree))))))
```

```
> (enumerate-interval 2 7)
(2 3 4 5 6 7)
> (enumerate-tree (list 1 (list 2 (list 3 4)) 5))
(1 2 3 4 5)
```

Lists as Conventional Interfaces (2)

```
(define (sum-odd-squares2 tree)
  (accumulate +
              0
              (map square
                   (filter odd?
                           (enumerate-tree tree)))))
(define (even-fibs2 n)
  (accumulate cons
              '()
              (filter even?
                      (map fib
                           (enumerate-interval 0 n)))))
```

More modular, more abstract

```
(define (list-fib-squares n)
  (accumulate cons
              '()
              (map square
                   (map fib
                        (enumerate-interval 0 n)))))
```

Construct a list of the squares of the first n+1 fibonacci numbers

And Reusable!

Nested Mappings

For some n, find all ordered pairs (i,j) where $1 \leq j < i \leq n$ and i+j prime.

```
> (accumulate append
              '()
              (map (lambda (i)
                     (map (lambda (j) (cons i j))
                          (enumerate-interval 1 (- i 1))))
                   (enumerate-interval 1 n)))
```

Typically a problem for 'nested loops'

Abstract it out

Nested Mappings (ctd.)

combination of mapping and accumulating is very common

```
(define (flatmap proc seq)
  (accumulate append '() (map proc seq)))

(define (prime-sum? pair)
  (prime? (+ (car pair) (cdr pair))))

(define (prime-sum-pairs n)
  (filter prime-sum?
          (flatmap (lambda (i)
                   (map (lambda (j) (cons i j))
                        (enumerate-interval 1 (- i 1))))
                   (enumerate-interval 1 n)))))
```

Nested Mappings: Another Example

Generate all permutations of a list L:
for each x in L: generate all permutations
of L-{x} and put x in front of each one.

```
(define (remove item sequence)
  (filter (lambda (x) (not (= x item)))
          sequence))

(define (permutations s)
  (if (null? s)
      (list '())
      (flatmap (lambda (x)
                 (map (lambda (p) (cons x p))
                      (permutations (remove x s))))
               s)))
```

A Fundamental Twist in our Story

Until now: data = numbers, booleans, strings, pairs
+ compound data = cons

but we need arbitrary symbols to represent the following:

This is **not** `(list a b c d)`

```
(a b c d)
(23 45 17)
((Norah 12) (Molly 9) (Anna 7) (Lauren 6) (Charlotte 4))
(* (+ 23 45) (+ x 9))
(define (fact n) (if (= n 1) 1 (* n (fact (- n 1)))))
```

Lists containing symbols can look just like the expressions of our language

Quoting

Use quote to identify lists and symbols that are to be treated as data objects rather than as expressions to be evaluated

semantic → syntactic

```
(quote <number>)      = <number>
(quote <string>)      = <string>
(quote <boolean>)     = <boolean>
```

self-evaluating expressions (not interesting)

symbols are **not** strings!

```
(quote <identifier>) = <symbol>
(quote (d0 … dn))    = (list (quote d0) … (quote dn))
```

interesting for things which have other meanings without quote

shorthand: `'d = (quote d)`

Examples

```
> (define a 1)
> (define b 2)
> (list a b)
(1 2)
> (list (quote a) (quote b))
(a b)
> (list (quote a) b)
(a 2)
> (quote 1)
1
> (quote (define a 1))
(define a 1)
> (car (quote (a b c)))
a
```

```
> (cdr (quote (a b c)))
(b c)
> (quote (car (quote (a b c))))
(car '(a b c))
> 'a
a
> '(a b c)
(a b c)
> '(1 2 3)
(1 2 3)
> (1 2 3)
⊕ procedure application: expected
procedure, given: 1; arguments were: 2 3
```

compound objects using printed representation for lists

Remember the evaluation rule for combinations

eq?

```
> (eq? 1 2)
#f
> (eq? 'apple 'apple)
#t
> (eq? "apple" 'apple)
#f
> (eq? (cons 'apple 'pear) (cons 'apple 'pear))
#f
> (define c (cons 'apple 'pear))
> (eq? c c)
#t
```

Membership test procedure

```
(define (memq item x)
  (cond ((null? x) #f)
        ((eq? item (car x)) x)
        (else (memq item (cdr x)))))
```

```
> (memq 'apple '(pear banana prune))
#f
> (memq 'apple '(x (apple sauce) y apple pear))
(apple pear)
```

Scheme is Homoiconic

Scheme uses the same syntax for data and procedures: nested lists with atoms as leaves. This is known as homoiconicity.

```
> (define (f x)
    (* x x))
> (define g '(define (f2 x) (* x x)))
> (f 10)
100
> (f2 10)
f2: undefined;
 cannot reference undefined identifier
> (interaction-environment)
#<namespace:0>
> (eval g (interaction-environment))
> (f2 10)
100
>
```

Programs that are about some domain are called base-level programs. Programs that are about programs are called meta-level programs. Homoiconicity is a key enabler for meta programming.

```
> (define program (cons 'define (cons 'x (cons 10 '()))))
> (eval program (interaction-environment))
> x
10
```

Symbol Manipulation: Example

one reduction rule per expression type

$$\frac{dc}{dx} = 0$$

$$\frac{dx}{dx} = 1$$

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

algebraic expression

variable

```
> (deriv '(+ x 3) 'x)
(+ 1 0)
> (deriv '(* x y) 'x)
(+ (* x 0) (* 1 y))
> (deriv '(* (* x y) (+ x 3)) 'x)
(+ (* (* x y) (+ 1 0)) (* (+ (* x 0) (* 1 y)) (+ x 3)))
```

Abstract Data

`(variable? e)`	Is e a variable?
`(same-variable? v1 v2)`	Are v1 and v2 the same variable?
`(sum? e)`	Is e a sum?
`(addend e)`	Addend of the sum e
`(augend e)`	Augend of the sum e
`(make-sum a1 a2)`	Construct the sum of *a1* and *a2*
`(product? e)`	Is e a product?
`(multiplier e)`	Multiplier of the product e
`(multiplicand e)`	Multiplicand of the product e
`(make-product m1 m2)`	Construct the product of m1 and m2

Symbolic Derivation Rules

The rules in Scheme

```
(define (deriv exp var)
  (cond
    ((number? exp) 0)
    ((variable? exp)
     (if (same-variable? exp var) 1 0))
    ((sum? exp)
     (make-sum (deriv (addend exp) var)
               (deriv (augend exp) var)))
    ((product? exp)
     (make-sum
      (make-product (multiplier exp)
                    (deriv (multiplicand exp) var))
      (make-product (deriv (multiplier exp) var)
                    (multiplicand exp))))
    (else
     (error "unknown expression type -- deriv" exp))))
```

Representing Expressions

```
(define (variable? e)
  (symbol? e))
(define (same-variable? v1 v2)
  (and (variable? v1) (variable? v2) (eq? v1 v2)))
```

```
(define (make-sum a1 a2)
  (list '+ a1 a2))
(define (sum? x)
  (and (pair? x) (eq? (car x) '+)))
(define (addend s)
  (cadr s))
(define (augend s)
  (caddr s))
```

```
(define (make-product m1 m2)
  (list '* m1 m2))
(define (product? x)
  (and (pair? x) (eq? (car x) '*)))
(define (multiplier p)
  (cadr p))
(define (multiplicand p)
  (caddr p))
```

Reducing the Answers

```
> (deriv '(* (* x y) (+ x 3)) 'x)
(+ (* (* x y) (+ 1 0)) (* (+ (* x 0) (* 1 y)) (+ x 3)))
```

Can be simplified

```
(define (=number? exp num)
  (and (number? exp) (= exp num)))

(define (make-sum a1 a2)
  (cond ((=number? a1 0) a2)
        ((=number? a2 0) a1)
        ((and (number? a1) (number? a2) (+ a1 a2)))
        (else (list '+ a1 a2))))
(define (make-product m1 m2)
  (cond ((or (=number? m1 0) (=number? m2 0)) 0)
        ((=number? m1 1) m2)
        ((=number? m2 1) m1)
        ((and (number? m1) (number? m2)) (* m1 m2))
        (else (list '* m1 m2))))
```

c.f. rationals

```
> (deriv '(* (* x y) (+ x 3)) 'x)
(+ (* x y) (* y (+ x 3)))
```

much better

Status So Far

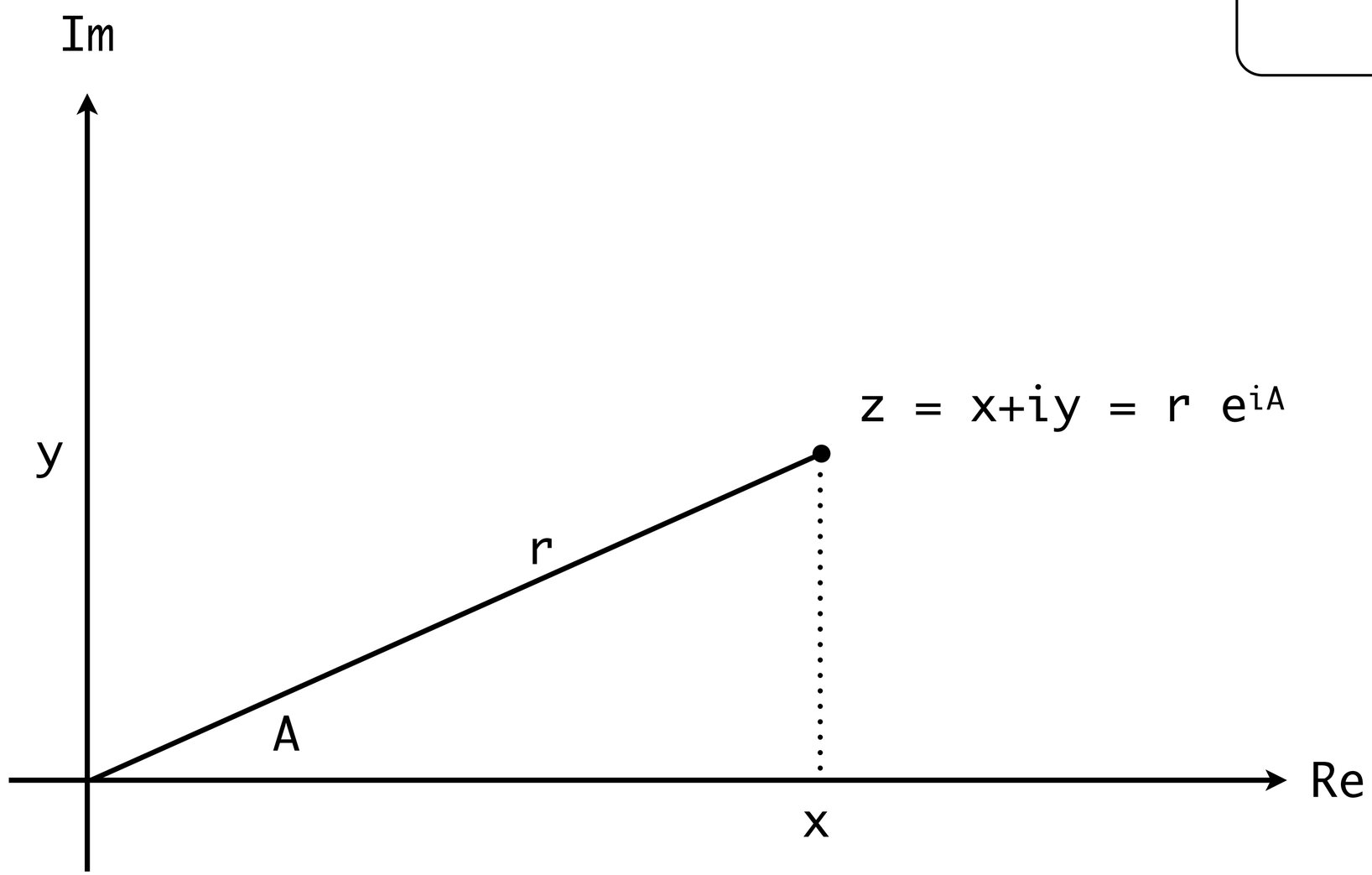
	data	procedures
primitive	`0, 1, -3.14, …`	`+, -, *, …`
combinations	`(cons numer denom)`	`(lambda (x) ...)`

abstraction	`(make-rat n d)`
	`(numer r)`
	`(denom r)`
		`(define f (lambda (x) ...)`

Now we treat the coexistence of multiple representations for abstract data

Let’s build a math-package

We will need complex numbers



$z = x+iy = r\ e^{iA}$

There are 2 possible representations

Architecture of the System

Programs that use complex numbers

`add-complex, sub-complex, mul-complex, div-complex`

Complex-arithmetic package

Rectangular representation	Polar representation

Make as much code as possible independent of the representation

List structure and primitive machine arithmetic

```
(make-from-real-imag (real-part z) (imag-part z)) = z

(make-from-mag-ang (magnitude z) (angle z)) = z
```

Laws satisfied by a representation

Code Independent of Representation

$(5 + 2i) + (3 - 7i) = 8 - 5i$

Some code makes more sense in polar notation; other in rectangular

```
(define (add-complex z1 z2)
  (make-from-real-imag (+ (real-part z1) (real-part z2))
                       (+ (imag-part z1) (imag-part z2))))
(define (sub-complex z1 z2)
  (make-from-real-imag (- (real-part z1) (real-part z2))
                       (- (imag-part z1) (imag-part z2))))
(define (mul-complex z1 z2)
  (make-from-mag-ang (* (magnitude z1) (magnitude z2))
                     (+ (angle z1) (angle z2))))
(define (div-complex z1 z2)
  (make-from-mag-ang (/ (magnitude z1) (magnitude z2))
                     (- (angle z1) (angle z2))))
```

$14e^{273i} * 4e^{3i} = 56e^{276i}$

Two Possible Representations

Rectangular

```
(define (real-part z) (car z))
(define (imag-part z) (cdr z))
(define (magnitude z)
  (sqrt (+ (square (real-part z)) (square (imag-part z)))))
(define (angle z)
  (atan (imag-part z) (real-part z)))
(define (make-from-real-imag x y) (cons x y))
(define (make-from-mag-ang r a)
  (cons (* r (cos a)) (* r (sin a))))
```

Polar

```
(define (real-part z)
  (* (magnitude z) (cos (angle z))))
(define (imag-part z)
  (* (magnitude z) (sin (angle z))))
(define (magnitude z) (car z))
(define (angle z) (cdr z))
(define (make-from-real-imag x y)
  (cons (sqrt (+ (square x) (square y)))
        (atan y x)))
(define (make-from-mag-ang r a) (cons r a))
```

`(cons x y) = ?`

$\Longrightarrow$ cannot coexist

Goal: Coexistence of Representations

```
(define (attach-tag type-tag contents)
  (cons type-tag contents))
(define (type-tag datum)
  (if (pair? datum)
      (car datum)
      (error "Bad tagged datum -- TYPE-TAG" datum)))
(define (contents datum)
  (if (pair? datum)
      (cdr datum)
      (error "Bad tagged datum -- CONTENTS" datum)))
```

Tagging of Data

In other words: make code as generic as possible.

Applied to Complex numbers

```
(define (rectangular? z)
  (eq? (type-tag z) 'rectangular))
(define (polar? z)
  (eq? (type-tag z) 'polar))
```

defer concrete representation to last possible moment (i,.e., in constructors and selectors)

data abstraction =
"principle of least commitment"

Rectangular Representation (2nd)

```
(define (real-part-rectangular z) (car z))
(define (imag-part-rectangular z) (cdr z))
(define (magnitude-rectangular z)
  (sqrt (+ (square (real-part-rectangular z))
           (square (imag-part-rectangular z)))))
(define (angle-rectangular z)
  (atan (imag-part-rectangular z)
        (real-part-rectangular z)))
(define (make-from-real-imag-rectangular x y)
  (attach-tag 'rectangular (cons x y)))
(define (make-from-mag-ang-rectangular r a)
  (attach-tag 'rectangular
              (cons (* r (cos a)) (* r (sin a)))))
```

append tag to representation procedures

untagged selectors (expect correct representation)

append tag to data

Polar Representation (2nd)

```
(define (real-part-polar z)
  (* (magnitude-polar z) (cos (angle-polar z))))
(define (imag-part-polar z)
  (* (magnitude-polar z) (sin (angle-polar z))))
(define (magnitude-polar z) (car z))
(define (angle-polar z) (cdr z))
(define (make-from-real-imag-polar x y)
  (attach-tag 'polar
              (cons (sqrt (+ (square x) (square y)))
                    (atan y x))))
(define (make-from-mag-ang-polar r a)
  (attach-tag 'polar (cons r a)))
```

Generic Representation

Type-based Dispatching

```
(define (real-part z)
  (cond ((rectangular? z)
         (real-part-rectangular (contents z)))
        ((polar? z)
         (real-part-polar (contents z)))
        (else (error "Unknown type -- REAL-PART" z))))
(define (imag-part z)
  (cond ((rectangular? z)
         (imag-part-rectangular (contents z)))
        ((polar? z)
         (imag-part-polar (contents z)))
        (else (error "Unknown type -- IMAG-PART" z))))
(define (magnitude z)
  (cond ((rectangular? z)
         (magnitude-rectangular (contents z)))
        ((polar? z)
         (magnitude-polar (contents z)))
        (else (error "Unknown type -- MAGNITUDE" z))))
(define (angle z)
  (cond ((rectangular? z)
         (angle-rectangular (contents z)))
        ((polar? z)
         (angle-polar (contents z)))
        (else (error "Unknown type -- ANGLE" z))))
```

Strip off tag and pass to representation implementation

Constructors Choose a Representation

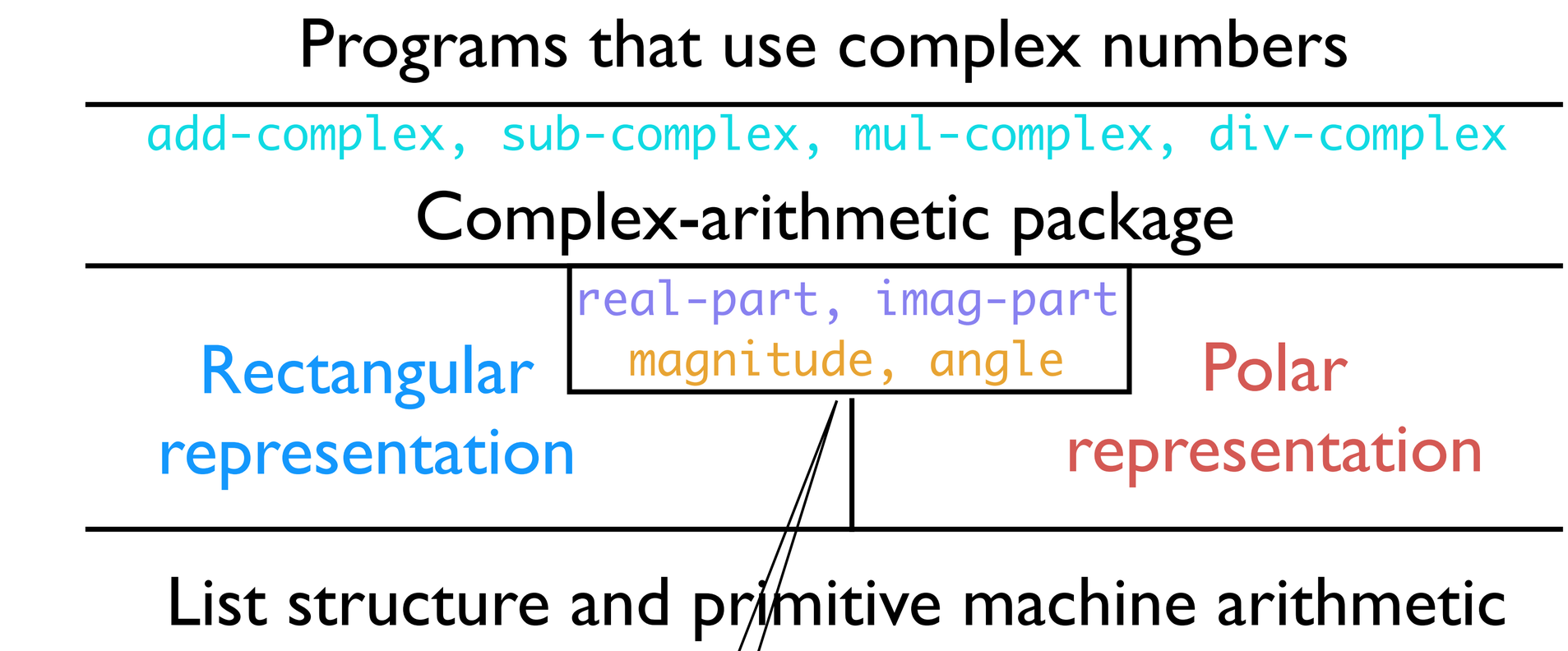
"Reasonable" Choice

```
(define (make-from-real-imag x y)
  (make-from-real-imag-rectangular x y))
(define (make-from-mag-ang r a)
  (make-from-mag-ang-polar r a))
```

unchanged!

```
(define (add-complex z1 z2)
  (make-from-real-imag (+ (real-part z1) (real-part z2))
                       (+ (imag-part z1) (imag-part z2))))
(define (sub-complex z1 z2)
  (make-from-real-imag (- (real-part z1) (real-part z2))
                       (- (imag-part z1) (imag-part z2))))
(define (mul-complex z1 z2)
  (make-from-mag-ang (* (magnitude z1) (magnitude z2))
                     (+ (angle z1) (angle z2))))
(define (div-complex z1 z2)
  (make-from-mag-ang (/ (magnitude z1) (magnitude z2))
                     (- (angle z1) (angle z2))))
```

Summary



Adding a representation requires us to hand-code all these again!

Let's fix this!

What we want

Representational Types

Operations	Polar	Rectangular
real-part	real-part-polar	real-part-rectangular
imag-part	imag-part-polar	imag-part-rectangular
magnitude	magnitude-polar	magnitude-rectangular
angle	angle-polar	angle-rectangular

Implementations

Let's make this table explicit. Adding a representation then merely requires us to extend the table.

"data-directed programming"

Let Us Assume Table Operations

Notice the “state change”!

```
(put <op> <type> <item>)

(get <op> <type>)
```

Implementation:
Chapter 3

Rectangular Code

```
(define (install-rectangular-package)
  ;; internal procedures
  (define (real-part z) (car z))
  (define (imag-part z) (cdr z))
  (define (make-from-real-imag x y) (cons x y))
  (define (magnitude z)
    (sqrt (+ (square (real-part z))
             (square (imag-part z)))))
  (define (angle z)
    (atan (imag-part z) (real-part z)))
  (define (make-from-mag-ang r a)
    (cons (* r (cos a)) (* r (sin a))))
  ;; interface to the rest of the system
  (define (tag x) (attach-tag 'rectangular x))
  (put 'real-part '(rectangular) real-part)
  (put 'imag-part '(rectangular) imag-part)
  (put 'magnitude '(rectangular) magnitude)
  (put 'angle '(rectangular) angle)
  (put 'make-from-real-imag 'rectangular
       (lambda (x y) (tag (make-from-real-imag x y))))
  (put 'make-from-mag-ang 'rectangular
       (lambda (r a) (tag (make-from-mag-ang r a))))
  'done)
```

Packages can have the same name for local procedures!

```
(put <op> <type> <item>)
```

Dispatch types are lists of arguments

Constructor-dispatch done on return type

Polar Code

```
(define (install-polar-package)
  ;; internal procedures
  (define (magnitude z) (car z))
  (define (angle z) (cdr z))
  (define (make-from-mag-ang r a) (cons r a))
  (define (real-part z)
    (* (magnitude z) (cos (angle z))))
  (define (imag-part z)
    (* (magnitude z) (sin (angle z))))
  (define (make-from-real-imag x y)
    (cons (sqrt (+ (square x) (square y)))
          (atan y x)))
  ;; interface to the rest of the system
  (define (tag x) (attach-tag 'polar x))
  (put 'real-part '(polar) real-part)
  (put 'imag-part '(polar) imag-part)
  (put 'magnitude '(polar) magnitude)
  (put 'angle '(polar) angle)
  (put 'make-from-real-imag 'polar
       (lambda (x y) (tag (make-from-real-imag x y))))
  (put 'make-from-mag-ang 'polar
       (lambda (r a) (tag (make-from-mag-ang r a))))
  'done)
```

Packages can have the same name for local procedures!

```
(put <op> <type> <item>)
```

Dispatch types are lists of arguments

Constructor-dispatch done on return type

Applying An Operation

```
(define (apply-generic op . args)
  (let ((type-tags (map type-tag args)))
    (let ((proc (get op type-tags)))
      (if proc
          (apply proc (map contents args))
          (error
            "No method for these types -- APPLY-GENERIC"
            (list op type-tags))))))
```

Determine type of all arguments

If a corresponding operation exists, then apply it

```
(define (real-part z) (apply-generic 'real-part z))
(define (imag-part z) (apply-generic 'imag-part z))
(define (magnitude z) (apply-generic 'magnitude z))
(define (angle z) (apply-generic 'angle z))
```

Implementation of the 4 accessors

```
(define (make-from-real-imag x y)
  ((get 'make-from-real-imag 'rectangular) x y))
(define (make-from-mag-ang r a)
  ((get 'make-from-mag-ang 'polar) r a))
```

Implementation of constructors

Data-driven Programming

Read Extension in the Book

Layered application of the technique

Programs that use numbers

`add sub mul div`

Generic Arithmetic package

`add-rat sub-rat`
`mul-rat div-rat`

Rational Arithmetic

`add-complex sub-complex`
`mul-complex div-complex`

Complex Arithmetic

Rectangular representation

Polar representation

`+ - * /`

Ordinary Arithmetic

List structure and primitive machine arithmetic

Data-driven Programming in Real-Life

A generic function is a "set" of methods

```
(defgeneric add (a b))

(defmethod add ((a number) (b number))
   (+ a b))
(defmethod add ((a vector) (b number))
   (map 'vector (lambda (n) (+ n b)) a))

(defmethod add ((a vector) (b vector))
   (map 'vector #'+ a b))
```

Data-driven programming is the basis of the Common Lisp Object System (CLOS), Julia, and Clojure.

Every method declares dynamic types for the args

The "multiple dispatch" happens at runtime

```
(add 2 3)                    ; returns 5
(add #(1 2 3 4) 7)           ; returns #(8 9 10 11)
(add #(1 2 3 4) #(4 3 2 1)) ; returns #(5 5 5 5)
```

This is similar (but very ≠) to Java/C++ overloading

Summary Chapters 1 and 2

	data	procedures
primitive	`0, 1, -3.14, …`	`+, -, *, …`
combinations	`(cons numer denom)`	`(lambda (x) ...)`
abstraction	`(make-rat n d)` `('<tag> <contents>)`	`(define f (lambda (x) ...))`

We can now write and abstract over generic procedures that operate on abstractions over multiple representations of data combinations

Chapter 2